Today 19.4, 19.5

$\boxed{L36}$

Today 19.4, 19.5

$\underbrace{19.4}_{\text{Forced Vibrations}}$

$\boxed{L36}$

Today 19.4, 19.5

$\boxed{L36}$

Forced Vibrations

Damped Vibrations

Today 19.4, 19.5

$\boxed{L36}$

Wednesday Review

Today 19.4, 19.5

$\boxed{L36}$

Wednesday Review

Friday Holiday

Today 19.4, 19.5

$\boxed{L36}$

Wednesday Review

Friday Holiday

Monday Nov. 30th Exam #4

L36

Today 19.4, 19.5

Wednesday Review

Friday Holiday

Monday Nov. 30th Exam #4

Wednesday Dec 2nd Day of Reckoning

Today 19.4, 19.5

L36

Wednesday Review

Friday Holiday

Monday Nov. 30th Exam #4

Wednesday Dec 2nd Day of Reckoning

Friday Dec 4th Final exam

Previously we saw that equations of the form $A\ddot{x} + Bx = \theta$, Previously we saw that equations of the form $A\ddot{x} + Bx = 0$, where A & B are Real and > 0

Previously we saw that equations of the form $A\ddot{x}+Bx=0$, where A & B are Real and >0 can be written as $\ddot{x}=-\omega_n^2 x$, Previously we saw that equations of the form $A\ddot{x}+Bx=0$, where A & B are Real and >0 can be written as $\ddot{x}=-\omega_n^2 x$, where $\omega_n=\sqrt{B/A}$

Previously we saw that equations of the form $A\ddot{x}+Bx=0$, where A & B are Real and >0 can be written as $\ddot{x}=-\omega_n^2 x$, where $\omega_n=\sqrt{B/A}$ and have solution

$$x=X_m\sin(\omega_n t+\varphi)$$

Previously we saw that equations of the form $A\ddot{x}+Bx=0$, where A & B are Real and >0 can be written as $\ddot{x}=-\omega_n^2 x$, where $\omega_n=\sqrt{B/A}$ and have solution

$$x=X_m\sin(\omega_n t+\varphi)$$

For an equation of the form

$$A\ddot{x}+Bx=C\sin(\omega_F t)$$

Previously we saw that equations of the form $A\ddot{x}+Bx=0$, where A & B are Real and >0 can be written as $\ddot{x}=-\omega_n^2 x$, where $\omega_n=\sqrt{B/A}$ and have solution

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For an equation of the form

$$A\ddot{x}+Bx=\underbrace{C\sin(\omega_F t)}_{\text{forcing term}}$$

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For an equation of the form $A\ddot{x}+Bx=\underbrace{C\sin(\omega_F t)}_{\text{forcing term}}$ we have two solutions

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* Homogeneous part

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$$x = X_m \sin(\omega_n t+\varphi)$$

For an equation of the form

$A\ddot{x}+Bx=\underline{C\sin(\omega_F t)}$ we have two solutions:

* Homogeneous part [forcing term $=0$]: $A\ddot{x}+Bx=0$

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$$\Rightarrow \quad -A\omega_F^2 X_m+BX_m=C$$

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$$\Rightarrow \quad -A\omega_F^2 X_m + B X_m = C \Rightarrow \quad X_m = \left[\frac{C}{B-A\omega_F^2}\right]\frac{(1/B)}{(1/B)}$$

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$$\Rightarrow X_m=\frac{(C/B)}{1-\frac{A}{B}\omega_F^2}\text{, But } \underline{\frac{B}{A}=\omega_n^2}$$

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$$\Rightarrow X_m=\frac{(C/B)}{1-\frac{A}{B}\omega_F^2},\ \text{But } \frac{B}{A}=\omega_n^2 \Rightarrow X_m=\frac{(C/B)}{1-\omega_F^2/\omega_n^2}$$

Damping:

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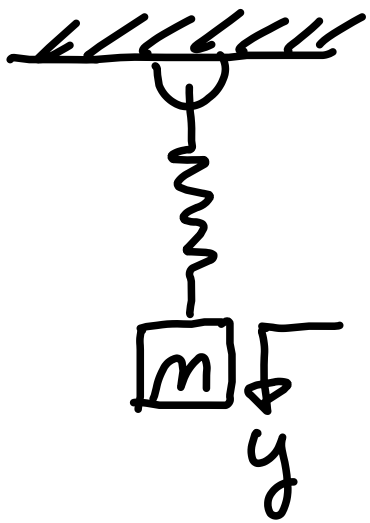

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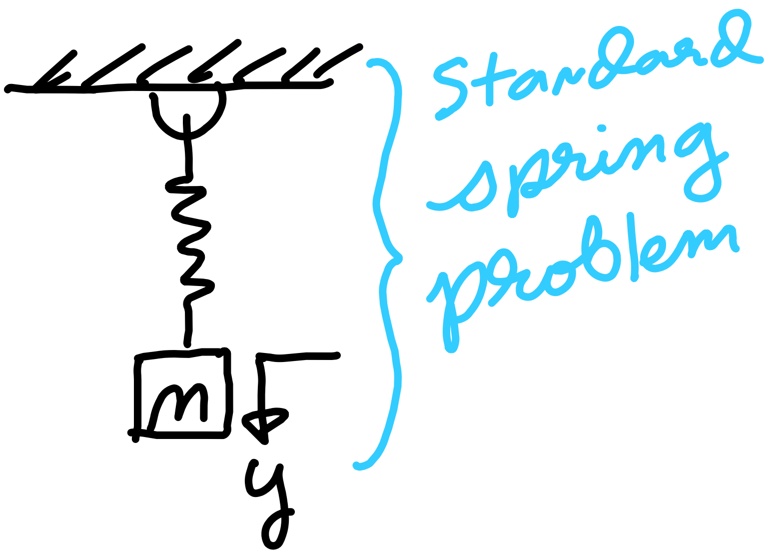

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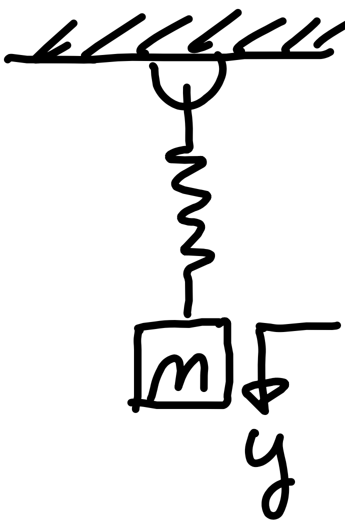
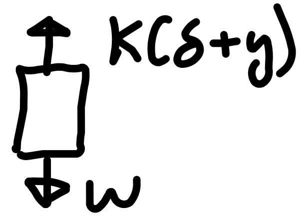


Standard spring problem



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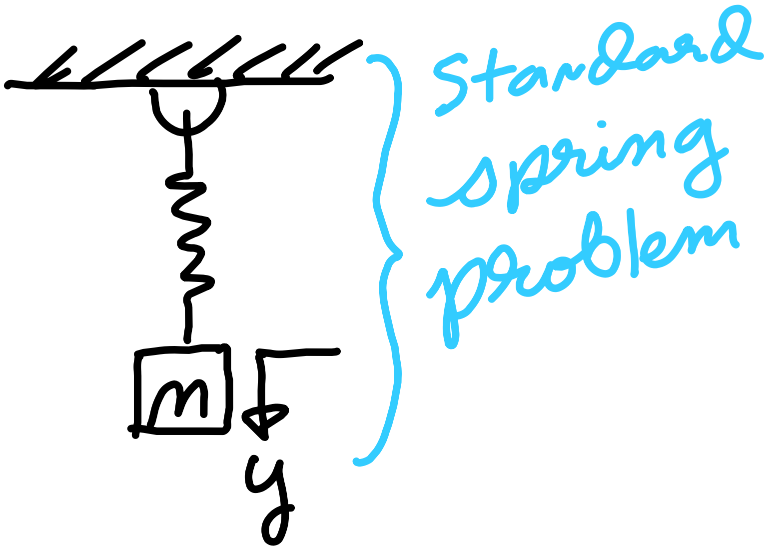

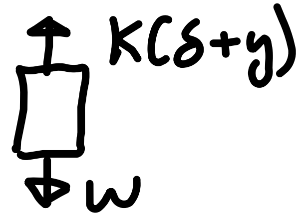

$\Sigma F_y = m\ddot{y} \Rightarrow$

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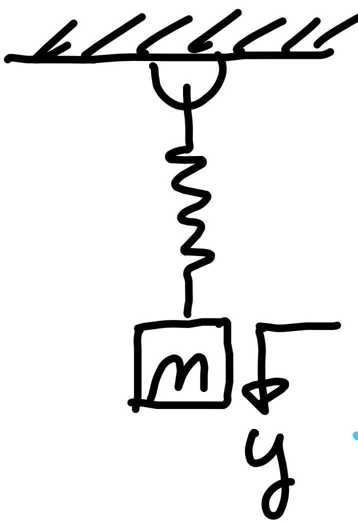

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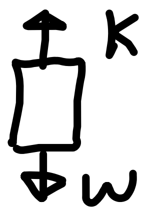

$$\Sigma F_y = m\ddot{y} \Rightarrow$$
$$-k(\delta+y)+w=m\ddot{y}$$

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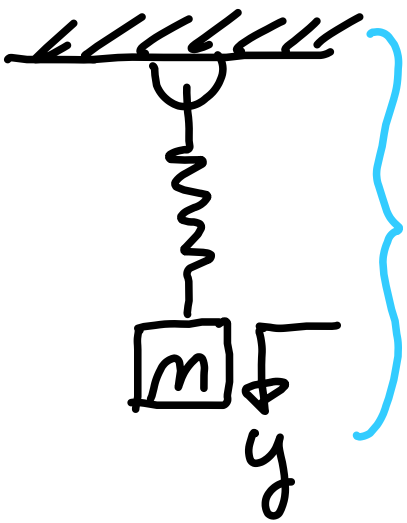

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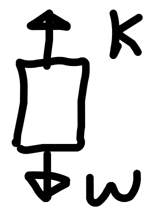

$$\Sigma F_y = m\ddot{y} \Rightarrow$$

$$-k(\delta+y)+w=m\ddot{y}$$

But $-k\delta+w=0$

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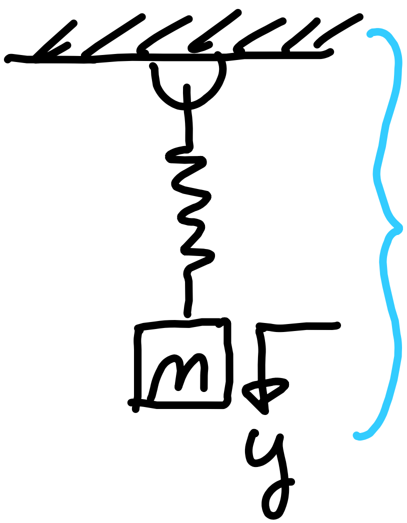

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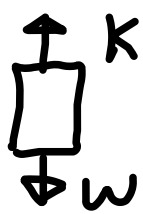

$$\Sigma F_y = m\ddot{y} \Rightarrow$$

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But $\underbrace{-k\delta+w=0}_{}$,

[Equilibrium]

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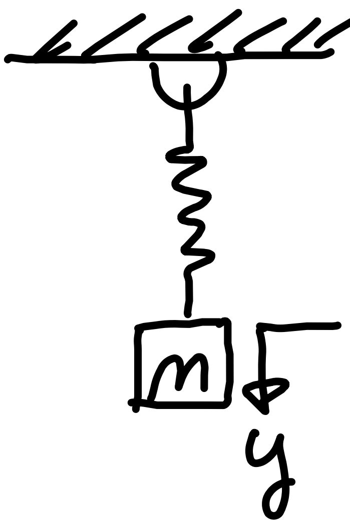

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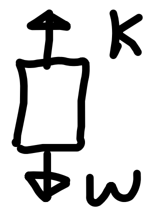


$$\Sigma F_y = m\ddot{y} \Rightarrow$$

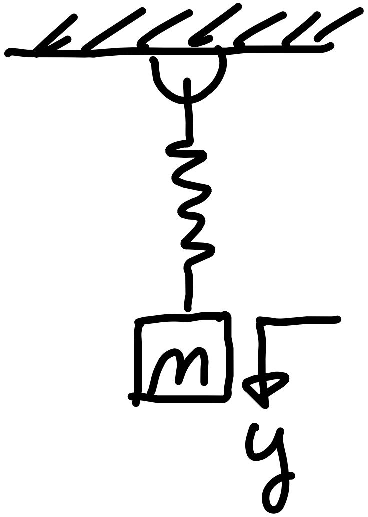
$$-k(\delta+y)+w=m\ddot{y}$$

But $\underline{-k\delta+w=0}$

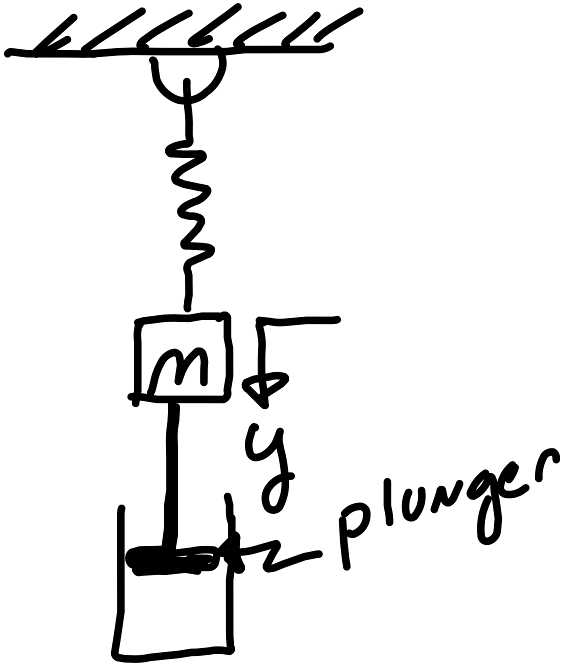
[Equilibrium]

so $m\ddot{y}+ky=0$

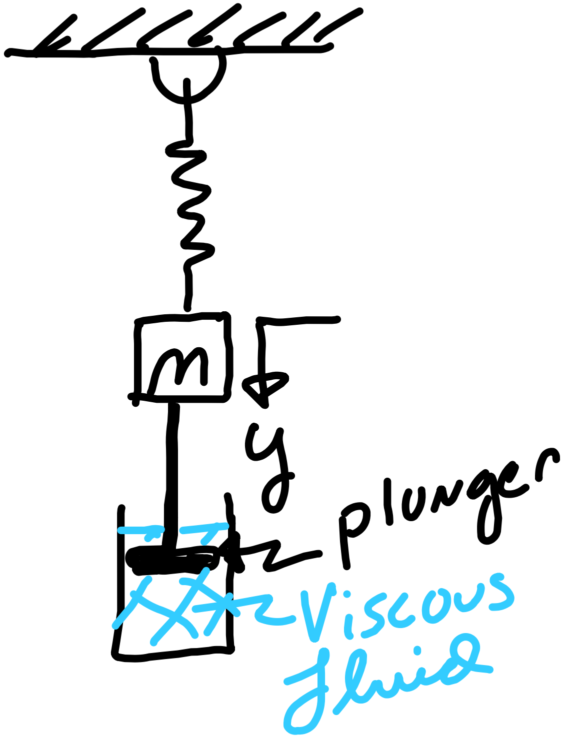
Now with Damping Force



Now with damping force

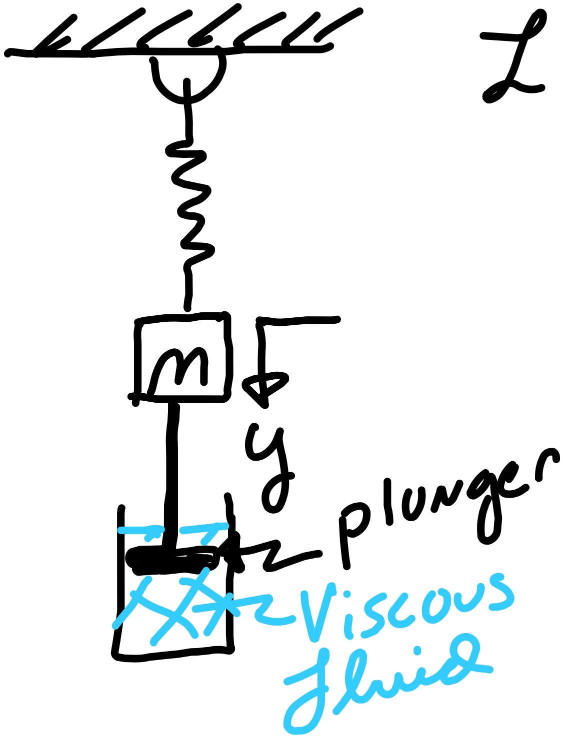


Now with Damping Force



Now with damping force

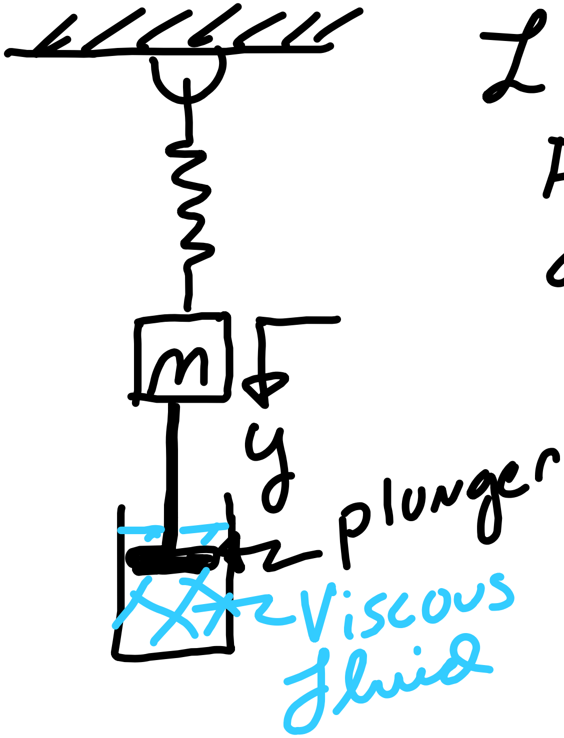
Let $F_d \equiv$ damping force



Now with Damping Force

Let $F_d \equiv$ damping force

$F_d = c\dot{y}$ that is in opposite direction to $\dot{y}$



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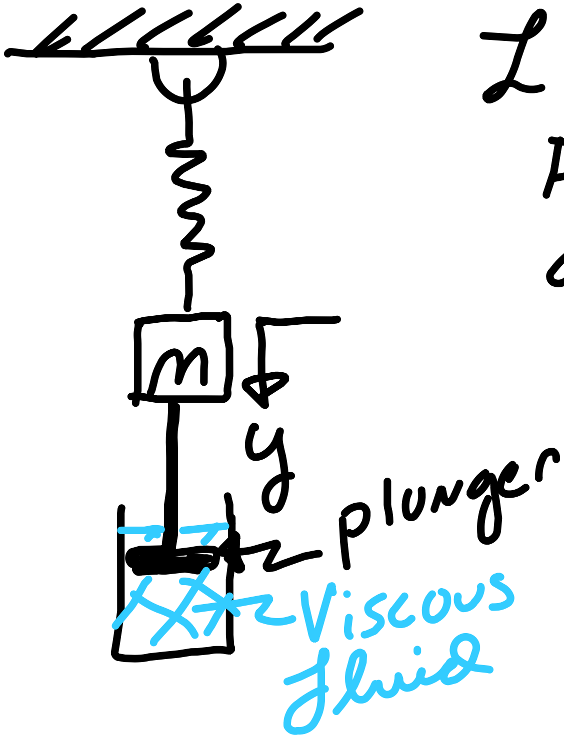

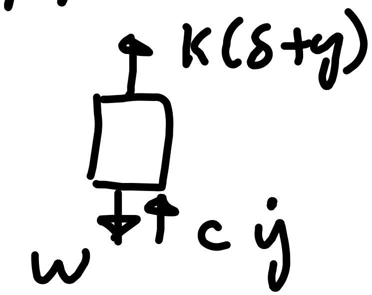

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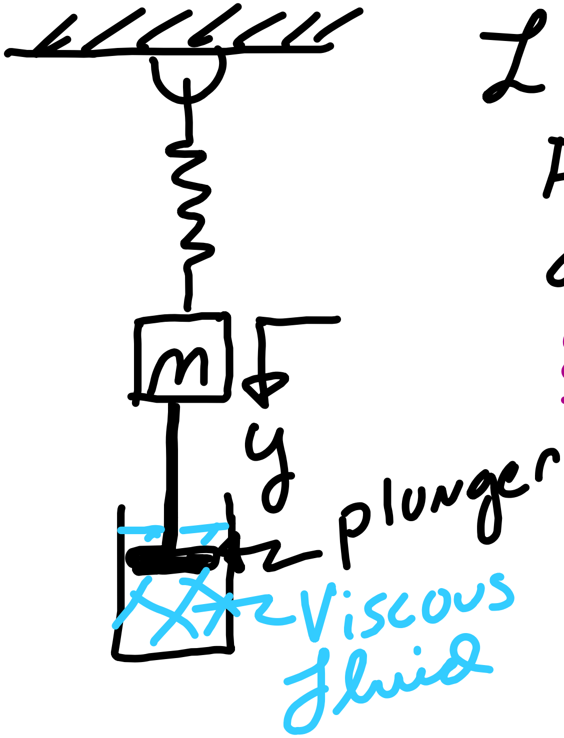
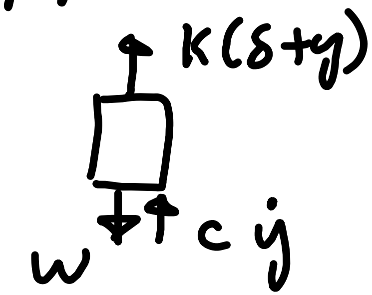
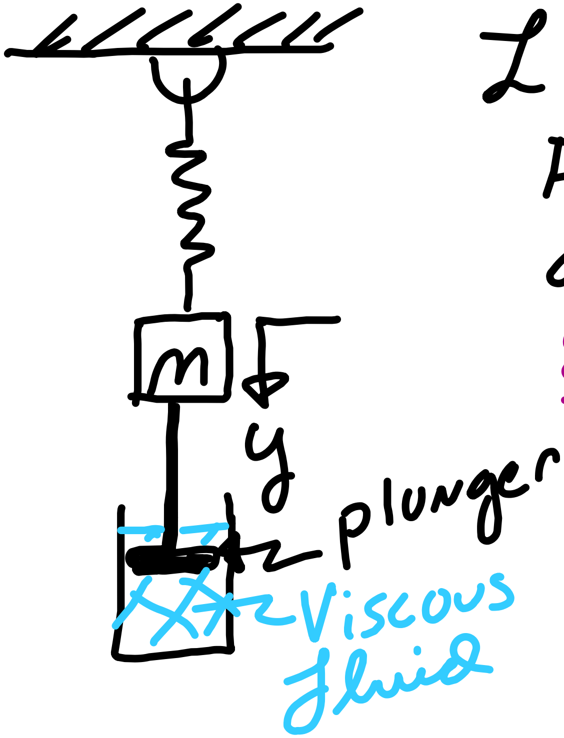




Equilibrium $\Sigma F_y = 0$

Now with damping force

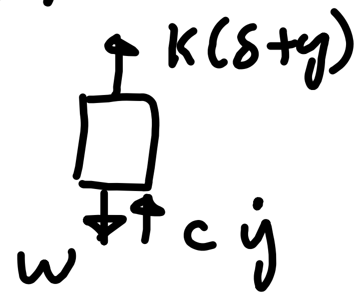


Let $F_d \equiv$ damping force

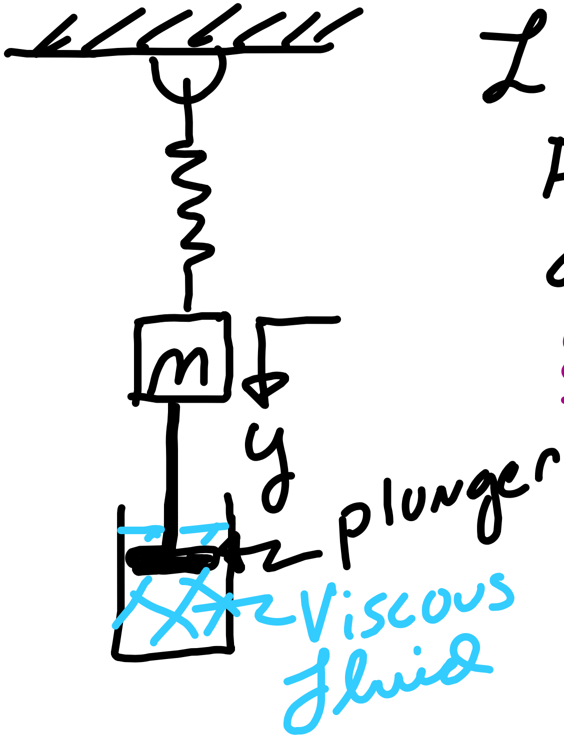
$F_d = c\dot{y}$ that is in opposite direction to $\dot{y}$

Equilibrium $\Sigma F_y = 0$

$\Rightarrow -K\delta + W = 0$



Now with damping force



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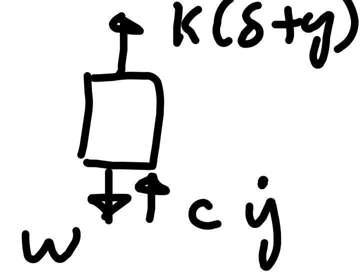
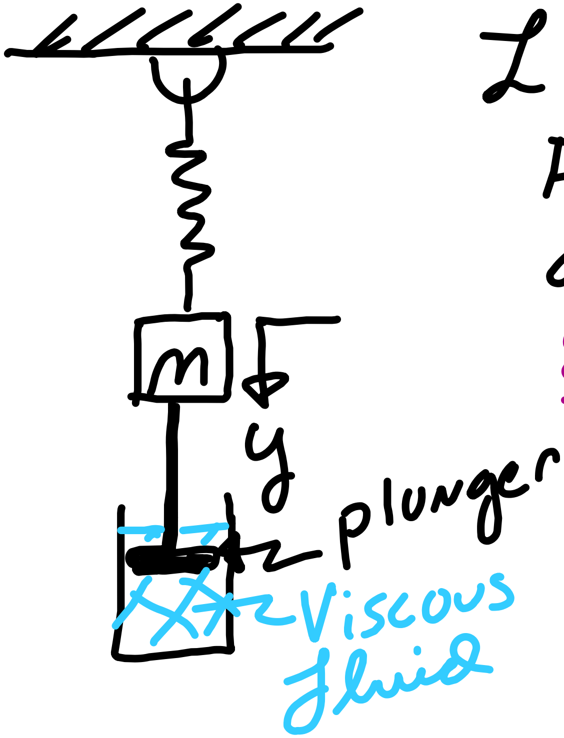


Equilibrium $\Sigma F_y = 0$

$\Rightarrow -k\delta + w = 0$

Out of equilibrium $\Sigma F_y = m\ddot{y} \Rightarrow$

Now with Damping Force



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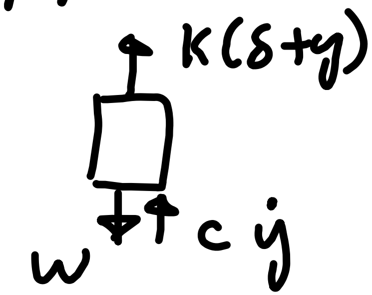


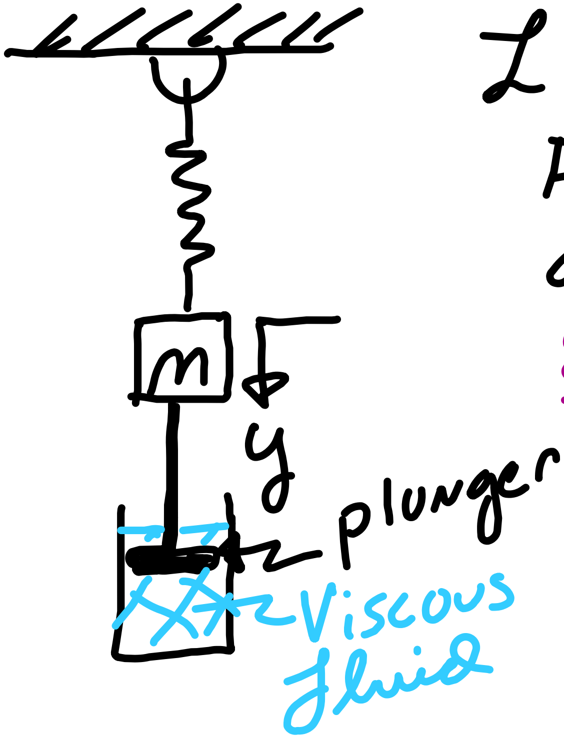
Equilibrium $\Sigma F_y = 0$

$$\Rightarrow -k\delta + w = 0$$

Out of equilibrium $\Sigma F_y = m\ddot{y} \Rightarrow$

$$-k(\delta+y) - c\dot{y} + w = m\ddot{y}$$

Now with damping force



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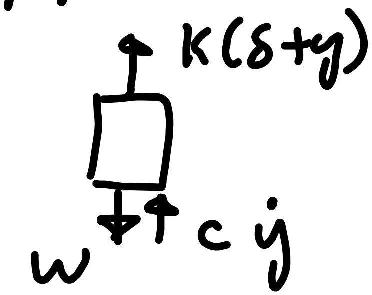


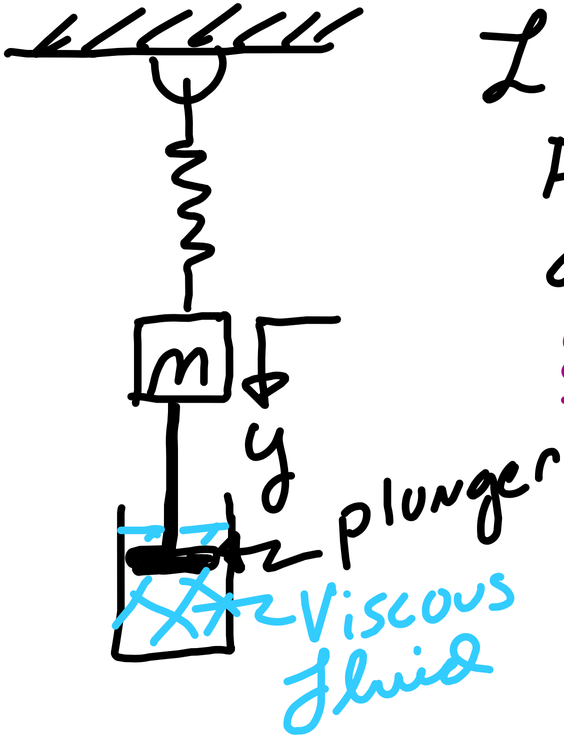
Equilibrium $\sum F_y = 0$

$\Rightarrow -k\delta + w = 0$

Out of equilibrium $\sum F_y = m\ddot{y} \Rightarrow$

$$-\underbrace{k(\delta}_{}+y) - c\dot{y} + \underbrace{w}_{} = m\ddot{y}$$

Now with damping force



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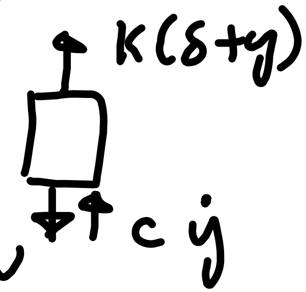


Equilibrium $\Sigma F_y = 0$

$\Rightarrow -k\delta + w = 0$

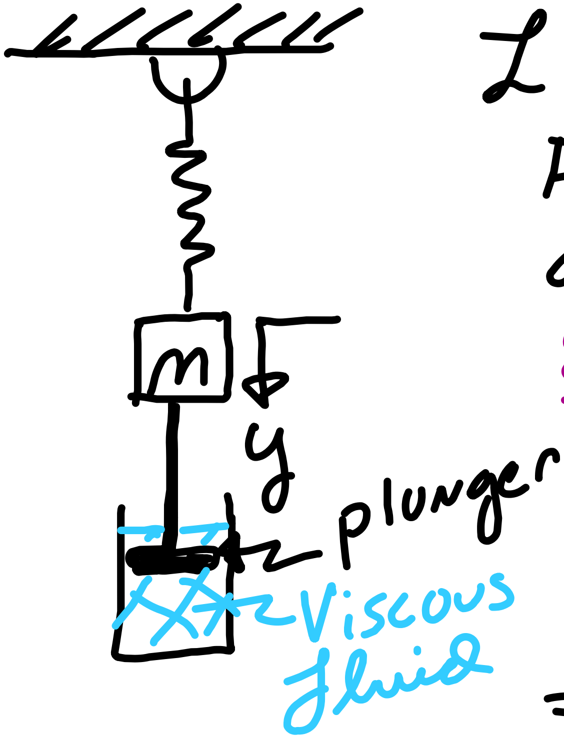
Out of equilibrium $\Sigma F_y = m\ddot{y} \Rightarrow$

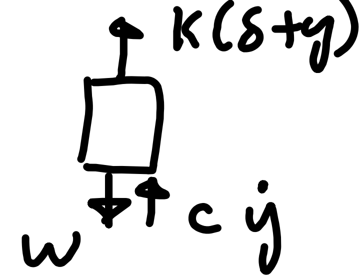
$-k(\delta + y) - c\dot{y} + w = m\ddot{y} \Rightarrow -ky - c\dot{y} = m\ddot{y}$

Now with damping force

Let $F_d \equiv$ damping force

$F_d = c\dot{y}$ that is in opposite direction to $\dot{y}$





Equilibrium $\Sigma F_y = 0$

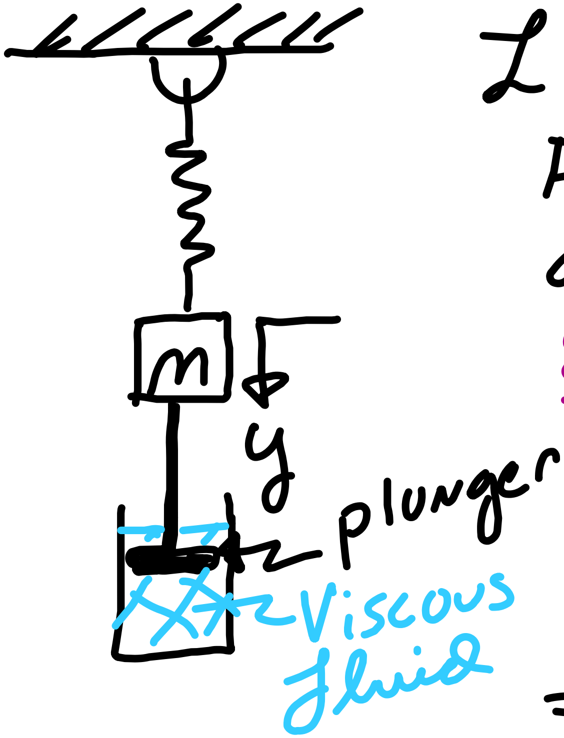
$\Rightarrow -k\delta + w = 0$

Out of equilibrium $\Sigma F_y = m\ddot{y} \Rightarrow$

$-\underbrace{k(\delta+y)} - c\dot{y} + \underbrace{w} = m\ddot{y} \Rightarrow -ky - c\dot{y} = m\ddot{y}$

$\Rightarrow \quad m\ddot{y} + c\dot{y} + ky = 0$

Now with Damping Force



Let $F_d \equiv$ damping force

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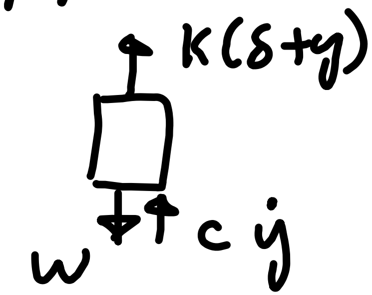

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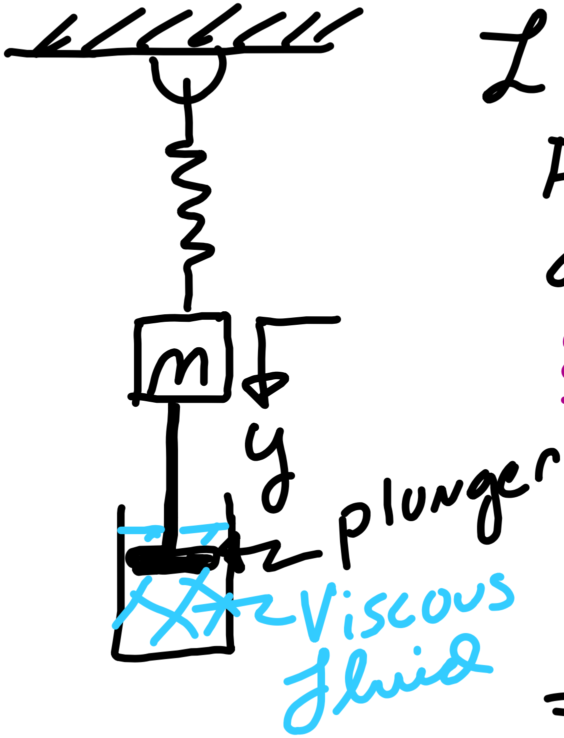
$-k(\delta+y) - c\dot{y} + w = m\ddot{y} \Rightarrow -ky - c\dot{y} = m\ddot{y}$

$\Rightarrow m\ddot{y} + c\dot{y} + ky = 0$. This is a second order homogeneous differential equation with Real coefficients.

Now with Damping Force



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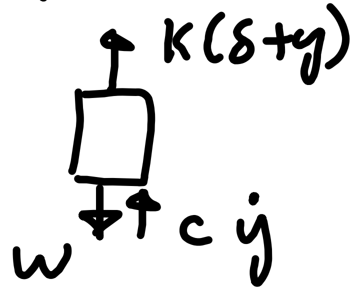

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$\Rightarrow \quad m\ddot{y} + c\dot{y} + ky = 0$. This is a second order homogeneous differential equation with Real coefficients. We solve such equations by assuming a solution of the form $y = e^{\lambda t}$

From previous slide

$$m\ddot{y}+c\dot{y}+ky=0$$

From previous slide

$m\ddot{y} + c\dot{y} + ky = 0$ & we take $y = e^{\lambda t}$

From previous slide

$m\ddot{y} + c\dot{y} + ky = 0$ & we take $y = e^{\lambda t}$

& since $\dot{y} = \lambda e^{\lambda t}$

From previous slide

$m\ddot{y} + c\dot{y} + ky = 0$ & we take $y = e^{\lambda t}$

& since $\dot{y} = \lambda e^{\lambda t}$ & $\ddot{y} = \lambda^2 e^{\lambda t}$

From previous slide

$m\ddot{y}+c\dot{y}+ky=0$ & we take $y=e^{\lambda t}$

& since $\dot{y}=\lambda e^{\lambda t}$ & $\ddot{y}=\lambda^2 e^{\lambda t}$ then

$$m\ddot{y}+c\dot{y}+ky=0 \Rightarrow m\lambda^2+c\lambda+k=0$$

From previous slide

$m\ddot{y}+c\dot{y}+ky=0$ & we take $y=e^{\lambda t}$

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$$m\ddot{y}+c\dot{y}+ky=0 \Rightarrow \underbrace{m\lambda^2+c\lambda+k=0}_{\text{characteristic equation}}$$

From previous slide

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The characteristic equation is a quadratic in variable λ,

From previous slide

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& since $\dot{y}=\lambda e^{\lambda t}$ & $\ddot{y}=\lambda^2 e^{\lambda t}$ then

$$m\ddot{y}+c\dot{y}+ky=0 \Rightarrow \underbrace{m\lambda^2+c\lambda+k=0}_{\text{characteristic equation}}$$

The characteristic equation is a quadratic in variable λ, so

$$\lambda=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

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<u>Note</u>: If $c = 0$ & $m > 0$ & $k > 0$

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Note: If $c = 0$ & $m > 0$ & $k > 0$ then

$$\lambda = \pm\sqrt{\frac{k}{m}}$$

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<u>Note</u>: If $c = 0$ & $m > 0$ & $k > 0$ then

$\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$,

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Note: If $c = 0$ & $m > 0$ & $k > 0$ then

$\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega$, with $\omega = \sqrt{\frac{k}{m}}$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

__Note__: If $c = 0$ & $m > 0$ & $k > 0$ then

$\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

So two solutions:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Note: If $c = 0$ & $m > 0$ & $k > 0$ then

$\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

So two solutions: $Ae^{+i\omega_n t}$ &

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

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So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

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So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

$\Rightarrow \quad x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$

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So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

$\Rightarrow \quad x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$

$$\lambda = \frac{-c \pm \sqrt{c^2-4km}}{2m}$$

Note: If $c=0$ & $m>0$ & $k>0$ then $\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

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$\Rightarrow \quad x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$ &

$\dot{x}(0) = 0$

$$\lambda = \frac{-c \pm \sqrt{c^2-4km}}{2m}$$

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So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

$\Rightarrow \quad x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$ & $\dot{x}(0) = 0$ then $A + B = x_m$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

<u>Note</u>: If $c = 0$ & $m > 0$ & $k > 0$ then

$\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

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$\Rightarrow$ $x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$ &

$\dot{x}(0) = 0$ then $A + B = x_m$ & $Ai\omega_n - Bi\omega_n = 0$

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$\Rightarrow \quad A = B$

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Note: If $c = 0$ & $m > 0$ & $k > 0$ then $\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

$\Rightarrow$ $x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$ & $\dot{x}(0) = 0$ then $A + B = x_m$ & $Ai\omega_n - Bi\omega_n = 0$

$\Rightarrow$ $A = B$ & $A = B = \frac{x_m}{2}$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Note: If $c = 0$ & $m > 0$ & $k > 0$ then

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$\Rightarrow$ $A = B$ & $A = B = \frac{x_m}{2}$ $\Rightarrow$ $x = \left(\frac{x_m}{2}\right)\left[e^{i\omega_n t} + e^{-i\omega_n t}\right]$

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<u>Note</u>: If $c = 0$ & $m > 0$ & $k > 0$ then $\lambda = \pm\sqrt{\frac{k}{m}}$ or $\lambda = \pm i\omega_n$, with $\omega_n = \sqrt{\frac{k}{m}}$

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But $e^{i\omega_n t} = \cos(\omega_n t) + i\sin(\omega_n t)$

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But $e^{i\omega_n t} = \cos(\omega_n t) + i\sin(\omega_n t)$ &

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$\Rightarrow$ $x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0) = x_m$ & $\dot{x}(0) = 0$ then $A + B = x_m$ & $Ai\omega_n - Bi\omega_n = 0$

$\Rightarrow$ $A = B$ & $A = B = \frac{x_m}{2}$ $\Rightarrow$ $x = \left(\frac{x_m}{2}\right)\left[e^{i\omega_n t} + e^{-i\omega_n t}\right]$

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$e^{-i\omega_n t} = \cos(\omega_n t) - i\sin(\omega_n t)$ so

$$x = x_m\cos(\omega_n t)$$

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So two solutions: $Ae^{+i\omega_n t}$ & $Be^{-i\omega_n t}$

$\Rightarrow$ $x = Ae^{i\omega_n t} + Be^{-i\omega_n t}$ & if $x(0)=x_m$ &

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But $e^{i\omega_n t} = \cos(\omega_n t) + i\sin(\omega_n t)$ &

$e^{-i\omega_n t} = \cos(\omega_n t) - i\sin(\omega_n t)$ so

$x = x_m\cos(\omega_n t) = x_m\sin(\omega_n t + \pi/2)$

ASU

Back to original problem

with $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

Back to original problem

with $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$. We will look at

3 cases : $\sqrt{c^2 - 4mk} = 0$

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3 cases: $\sqrt{c^2 - 4mk} = 0$, $\sqrt{c^2 - 4mk} \in$ Reals

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3 cases: $\sqrt{c^2-4mk} = 0$, $\sqrt{c^2-4mk} \in$ Reals

& $\sqrt{c^2-4mk} \in$ Complex

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We will hit the important points now

Back to original problem

with $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$. We will look at

3 cases: $\sqrt{c^2-4mk} = 0$, $\sqrt{c^2-4mk} \in$ Reals & $\sqrt{c^2-4mk} \in$ Complex

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We will hit the important points now & leave the rest of the math for later pages of these lecture notes.

CASE I :

CASE I : $\sqrt{c^2-4mk} = 0$

$\underline{\underline{\text{CASE I}}}$: $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk$

CASE I : $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk}$

CASE I : $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk}$ & $\lambda = -\frac{c}{2m}$

CASE I : $\sqrt{c^2-4mk} = 0 \Rightarrow$

$$c^2 = 4mk \Rightarrow c = 2\sqrt{mk} \quad \& \quad \lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$$

$\underline{\underline{\text{CASE I}}}$: $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk} \;\;\$\;\; \lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$

But $\sqrt{\frac{k}{m}}$ = ω

CASE I : $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk}$ & $\lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$

But $\sqrt{\frac{k}{m}}$ = ωn so λ = -ωn.

CASE I: $\sqrt{c^2 - 4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow c = 2\sqrt{mk}$ & $\lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$

But $\sqrt{\frac{k}{m}} = \omega_n$ so $\lambda = -\omega_n$.

General solution for this case is

$$y = (A_1 + A_2 t)e^{-\omega_n t}$$

$\underline{\underline{\text{CASE I}}}$: $\sqrt{c^2-4mk} = \theta \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk}$ & $\lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$

But $\sqrt{\frac{k}{m}} = \omega_n$ so $\lambda = -\omega_n$.

General solution for this case is

$$y = (A_1 + A_2 t)e^{-\omega_n t}$$

Gets to equilibrium in shortest possible time!

CASE I : $\sqrt{c^2-4mk} = 0 \Rightarrow$

$c^2 = 4mk \Rightarrow \quad c = 2\sqrt{mk}$ & $\lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$

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Gets to equilibrium in shortest possible time !

& No vibration !

CASE I: $\sqrt{c^2-4mk} = 0 \Rightarrow$

$$c^2 = 4mk \Rightarrow c = 2\sqrt{mk} \;\;\&\;\; \lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}}$$

But $\sqrt{\frac{k}{m}} = \omega_n$ so $\lambda = -\omega_n$.

General solution for this case is

$$y = (A_1 + A_2 t)e^{-\omega_n t}$$

Gets to equilibrium in shortest possible time! & No vibration! & called

"Critically damped"

CASE II :

CASE II : $\sqrt{c^2 - 4mk} \in \mathbb{R}$

CASE II : $\sqrt{c^2-4mk} \in \mathbb{R} \Rightarrow$

$c^2 > 4mk$

$\underline{\underline{\text{CASE II}}}$: $\sqrt{c^2-4mk} \in \mathbb{R} \Rightarrow$

$c^2 > 4mk$. General solution

$$y = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_2 t}.$$

CASE II: $\sqrt{c^2-4mk} \in \mathbb{R} \Rightarrow$

$c^2 > 4mk$.

General solution

$$y = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_2 t}.$$

No vibration

CASE II : $\sqrt{c^2 - 4mk} \in \mathbb{R} \Rightarrow$

$c^2 > 4mk$. General solution

$$y = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_2 t}.$$

No vibration &

called

"Overdamped".

CASE III :

CASE III : $\sqrt{c^2-4mk} \in \mathbb{C}$

CASE III: $\sqrt{c^2-4mk} \in \mathbb{C} \Rightarrow$

$c^2 < 4mk$

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$c^2 < 4mk$ General solution

$$y = A\sin(\omega_d t + \phi)\, e^{-(c/2m)t}$$

<u>CASE III</u> : $\sqrt{c^2-4mk} \in \mathbb{C} \Rightarrow$

$c^2 < 4mk$ General solution

$$y = A\sin(\omega_d t + \phi)\, e^{-(c/2m)t}$$

<u>Vibrates</u>

CASE III : $\sqrt{c^2-4mk} \in \mathbb{C} \Rightarrow$

$c^2 < 4mk$ General solution

$$y = A\sin(\omega_d t + \varphi)\,e^{-(c/2m)t}$$

Vibrates & called "under damped"

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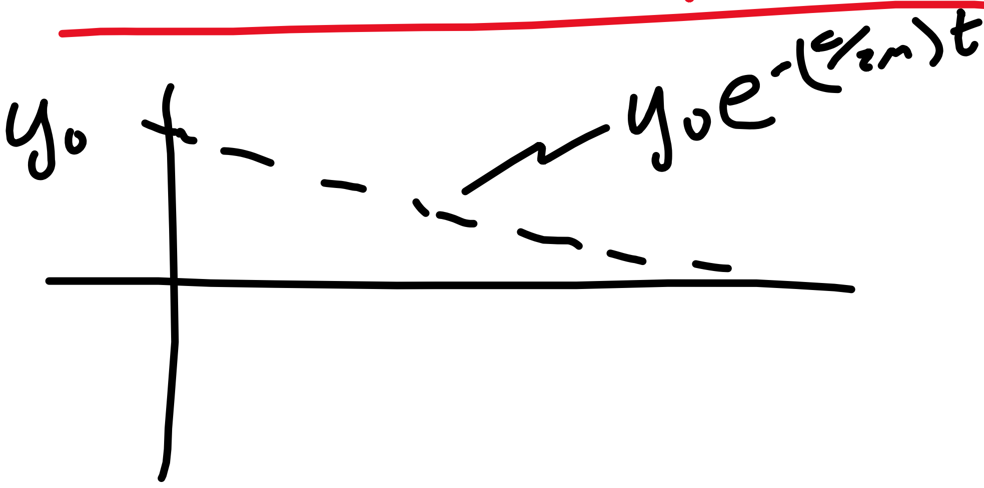

CASE III : $\sqrt{c^2-4mk} \in \mathbb{C} \Rightarrow$

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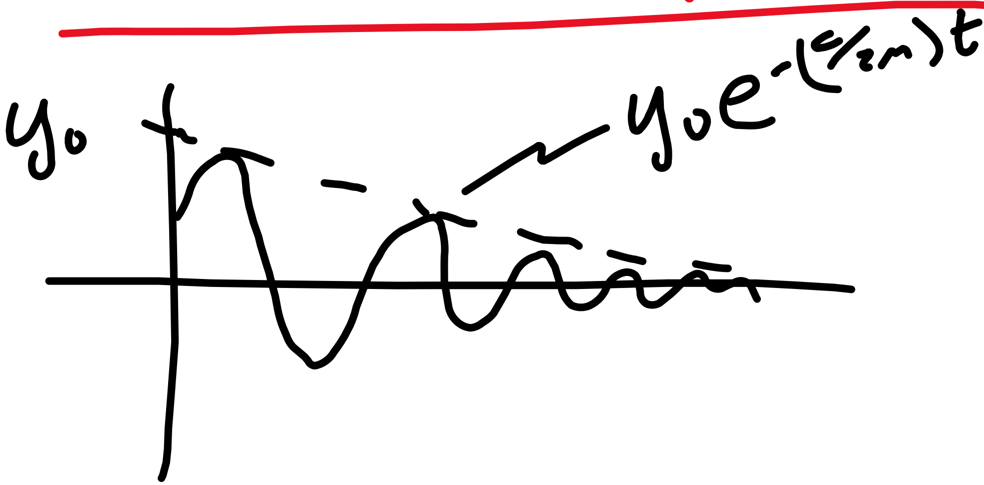
Vibrates & called "under damped"



Additional math:

CASE I: $\sqrt{c^2-4mk} = 0 \Rightarrow c = 2\sqrt{mk}$

$\Rightarrow \lambda = -\frac{c}{2m} = -\sqrt{\frac{k}{m}} = -\omega_n$ has solution $y = A_1 e^{-\omega_n t}$, where A_1 is some constant. Problem: We have a 2^{nd} order differential equation but only a single solution! Turns out we can take $y = A_2 t e^{-\omega_n t}$ as 2^{nd} solution. To check that this is true, we just need to show that $m\ddot{y} + c\dot{y} + ky = 0$

& since $\frac{d}{dt}(te^{-\omega_n t}) = e^{-\omega_n t} - t\omega_n e^{-\omega_n t}$ &

since $\frac{d^2}{dt^2}(te^{-\omega_n t}) = -\omega_n e^{-\omega_n t} - \omega_n e^{-\omega_n t} + t\omega_n^2 e^{-\omega_n t}$

$$= e^{-\omega_n t}(t\omega_n^2 - 2\omega_n)$$

then $m\frac{d^2}{dt^2}(te^{-\omega_n t}) + c\frac{d}{dt}(te^{-\omega_n t}) + kte^{-\omega_n t}$

$= (me^{-\omega_n t})(t\omega_n^2 - 2\omega_n) + (ce^{-\omega_n t})(1 - t\omega_n) + kte^{-\omega_n t}$

$= (e^{-\omega_n t})[t(m\omega_n^2 - c\omega_n + k) + (c - 2m\omega_n)]$ But $c = 2\sqrt{mk}$ & $\omega_n = \sqrt{\frac{k}{m}}$

$= (e^{-\omega_n t})[t(m\frac{k}{m} - 2\sqrt{mk}\sqrt{\frac{k}{m}} + k) + (2\sqrt{mk} - 2m\sqrt{\frac{k}{m}})]$

$= (e^{-\omega_n t})[t(k - 2k + k) + 2\sqrt{mk} - 2\sqrt{mk}] = 0$

So now we have 2 linearly independent solutions & can write the general solution as

$$y = (A_1 + A_2 t)e^{-\omega_n t}$$

CASE II : $\sqrt{c^2-4mk} \in \mathbb{R} \Rightarrow c > 2\sqrt{mk}$

Now $\lambda_1 = -\frac{c}{2m} + \frac{\sqrt{c^2-4mk}}{2m}$ & $\lambda_2 = -\frac{c}{2m} - \frac{\sqrt{c^2-4mk}}{2m}$

so general solution is $y = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

CASE III : $\sqrt{c^2-4mk} \in \mathbb{C} \Rightarrow c < 2\sqrt{mk}$

Now $\lambda_1 = -\frac{c}{2m} + \frac{\sqrt{c^2-4mk}}{2m}$ & $\lambda_2 = -\frac{c}{2m} - \frac{\sqrt{c^2-4mk}}{2m}$

& since $4mk > c^2$, can write as $\lambda_1 = -\frac{c}{2m} + i\frac{\sqrt{4mk-c^2}}{2m}$

& $\lambda_2 = -\frac{c}{2m} - i\frac{\sqrt{4mk-c^2}}{2m}$ or $\lambda_1 = -\frac{c}{2m} + i\omega_d$

& $\lambda_2 = -\frac{c}{2m} - i\omega_d$, where $\omega_d^2 = \frac{4mk-c^2}{4m^2}$

$\Rightarrow \omega_d^2 = \frac{k}{m} - \frac{c^2}{4m^2} = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$ Note: $\frac{k}{m} = \omega_n^2$

so $\omega_d^2 = \omega_n^2 - \left(\frac{c}{2m}\right)^2 \Rightarrow \omega_d < \omega_n \Rightarrow \tau_d > \tau_n$

General solution is $y = A_1 e^{-\frac{c}{2m}t + i\omega_d t} + A_2 e^{-\frac{c}{2m}t - i\omega_d t}$

$\Rightarrow y = \left[e^{-\left(\frac{c}{2m}\right)t}\right]\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$

$= \left[e^{-\left(\frac{c}{2m}\right)t}\right]\left[(A_1+A_2)\cos(\omega_d t) + i(A_1-A_2)\sin(\omega_d t)\right]$

& since $y \in \mathbb{R}$ then we can take

$A_1 + A_2 \equiv D_1$ & $i(A_1 - A_2) \equiv D_2$, where D_1 & $D_2 \in \mathbb{R}$.

Now $y = \left[e^{-\left(\frac{c}{2m}\right)t}\right]\left[D_1 \cos(\omega_d t) + D_2 \sin(\omega_d t)\right]$

or $y = y_0 e^{-\left(\frac{c}{2m}\right)t} \sin(\omega_d t + \phi)$

Connections between exponentials & trigs & hyperbolic trigs

Series expansions: $f(\theta) = f(0) + \frac{f'(0)}{1!}\theta + \frac{f''(0)}{2!}\theta^2 + \cdots$

$\sin(0) = 0$, $\frac{d}{d\theta}\sin\theta\Big|_{\theta=0} = \cos\theta\Big|_{\theta=0} = 1$

$\frac{d^2}{d\theta^2}(\sin\theta)\Big|_{\theta=0} = -\sin\theta\Big|_{\theta=0} = 0$, $\frac{d^3}{d\theta^3}(\sin\theta)\Big|_{\theta=0} = -\cos\theta\Big|_{\theta=0} = -1$, & on & on

$$\Rightarrow \boxed{\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots}$$

Similarly $\boxed{\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots}$

& $e^{i0} = 1$, $\frac{de^{i\theta}}{d\theta}\Big|_{\theta=0} = ie^{i\theta}\Big|_{\theta=0} = i$, $\frac{d^2e^{i\theta}}{d\theta^2}\Big|_{\theta=0} = -e^{i\theta}\Big|_{\theta=0} = -1$

$\frac{d^3e^{i\theta}}{d\theta^3} = -ie^{i\theta}\Big|_{\theta=0} = -i$, $\frac{d^4e^{i\theta}}{d\theta^4}\Big|_{\theta=0} = e^{i\theta}\Big|_{\theta=0} = 1$ & on & on

$$\Rightarrow \boxed{\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta \end{aligned}}$$

or $\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ \& } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$

note $\cos(i\theta) = \frac{e^{-\theta} + e^{\theta}}{2}$ & $\sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2i}$

But $\cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$ & $\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$

so $\boxed{\cosh(\theta) = \cos(i\theta) \text{ \& } \sinh(\theta) = -i\sin(i\theta)}$